

A Lie Group Structure for Fourier Integral Operators

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1. Introduction

In paper [1] we began discussing the problem of giving a Lie group structure to the group of invertible formal (i.e. classical and modulo smoothing) Fourier integral operators on a compact manifold M in such a way that the formal pseudodifferential operators become the Lie algebra. We outlined the procedure for doing this as follows.

Let $\mathcal{D}_\theta(T^*M \setminus O)$ denote the group of smooth diffeomorphisms of $T^*M \setminus O$ which preserve the canonical one-form θ . Let $(\text{FIO})_*$ denote the group of invertible formal Fourier integral operators on M and let $(\Psi DO)_*$ be the group of invertible formal pseudodifferential operators on M . Then we have an exact sequence of groups

$$I \longrightarrow (\Psi DO)_* \xrightarrow{j} (\text{FIO})_* \xrightarrow{\pi} \mathcal{D}_\theta(T^*M \setminus O) \longrightarrow e, \quad (1.1)$$

where j is inclusion and $\pi(A) = \eta$ if $\text{graph}(\eta) \subset (T^*M \setminus O) \times (T^*M \setminus O)$ is the canonical relation associated to A . The technique we use to give a Lie group

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structure to the group in the middle of this sequence is first to give Lie group structures to each of the groups at the ends of the sequence and then to construct a local section of the fibration π satisfying compatibility conditions that allow us to define charts on $(\text{FIO})_*$ which make the group operations C^∞ .

In infinite dimensions there are various types of manifold structures to choose from. For our purposes we find it most convenient to work in terms of ILH manifolds as described by Omori [15]. To do this we have to restrict attention to $(\text{FIO}_{0,k})_*$, the group of invertible formal Fourier integral operators of order zero modulo those of order $-k-1$. This gives a new exact sequence

$$I \longrightarrow (\Psi\text{DO}_{0,k})_* \xrightarrow{j} (\text{FIO}_{0,k})_* \xrightarrow{\pi} \mathcal{D}_\theta(T^*M \setminus O) \longrightarrow e, \quad (1.2)$$

where $(\Psi\text{DO}_{0,k})_*$ is the group of formal pseudodifferential operators of order zero modulo those of order $-k-1$. It was shown in [18] that the space $\mathcal{D}_\theta(T^*M \setminus O)$ is an ILH Lie group and in [1] that the space $(\Psi\text{DO}_{0,k})_*$ is an ILH Lie group. Furthermore in [1] a local section σ of the fibration $\pi: (\text{FIO}_{0,k})_* \rightarrow \mathcal{D}_\theta(T^*M \setminus O)$ was constructed in a neighborhood of the identity. In Sect. 2 of this paper we recall the basic definitions and facts from [18] and [1].

The main object of this paper is to show that the necessary compatibility conditions on σ are satisfied so that $(\text{FIO}_{0,k})_*$ can be made into an ILH Lie group. In Sect. 3 we formulate these compatibility conditions by looking at the abstract situation, i.e. an exact sequence

$$I \longrightarrow \mathcal{H} \xrightarrow{j} \mathcal{G} \xrightarrow{\pi} \mathcal{Q} \longrightarrow e \quad (1.3)$$

of groups with \mathcal{H} and \mathcal{Q} ILH Lie groups, and ask for a construction that will give \mathcal{G} an ILH Lie group structure that makes the maps in (1.3) smooth. As remarked above, what is needed is a local section $\sigma: \mathcal{U} \subset \mathcal{Q} \rightarrow \mathcal{G}$ which satisfies certain conditions of compatibility with the group structures and topologies. The idea is to identify $\pi^{-1}(\mathcal{U})$ with $\mathcal{U} \times \mathcal{H}$ by means of σ , and give the ILH manifold structure to \mathcal{G} by right translating this around. Overlap conditions and group compatibility conditions put requirements on σ .

In Sect. 4 we apply the abstract conditions found in Sect. 3 to the exact sequence (1.2) to obtain an ILH Lie group structure for $(\text{FIO}_{0,k})_*$. In Sect. 5 we comment on the Lie group structure of the whole group $(\text{FIO})_*$ of invertible formal Fourier integral operators and discuss the possibility of extending these techniques to give a Lie group structure to the full group of invertible Fourier integral operators (not just the formal ones).

In the series of papers [16], Omori et al. give the space of invertible Fourier integral operators of order zero the structure of a topological group. This topology is compatible with the topologies we give the spaces $(\text{FIO}_{0,k})_*$ in the sense that $(\text{FIO}_{0,k})_*$ is a quotient of this group (our charts are essentially the same as their charts except that we truncate the symbol). In [16] it is also promised that future papers will give a regular Fréchet-Lie group structure to the space of invertible Fourier integral operators of order zero¹. Our methods are quite different from

¹ While this paper was in final state of preparation for publication Omori et al. did proof it [16a, VIII]. Our work was available since August 1984 as MSRI (Berkeley) – preprint # 049-84-7, and was done without knowledge of theirs

those of [16] and stress the ILH-Lie group structure of the spaces $(\text{FIO}_{0,k})_*$. Using this structure one can give a Fréchet-Lie group structure to $(\text{FIO}_0)_*$ as the direct limit of the ILH-Lie groups $(\text{FIO}_{0,k})_*$ and then, if one wishes, the smoothing operators can be tacked on as in [16] (see Remark 1 of Sect. 5 in this paper, one still needs the argument of [16] to see that this is a topological group). Our point of view is that in implicit function type applications the ILH structure will be more useful than the Fréchet manifold structure (see comments in Sect. 2 on this matter).

2. Facts About Pseudodifferential and Fourier Integral Operators, ILH Lie Groups, and $\mathcal{D}_\theta(T^*M \setminus O)$

2.1. Formal Pseudodifferential and Fourier Integral Operators

For a compact manifold M let $I^m(M, C)$ denote the space of Fourier integral operators of order m on M associated with the homogeneous canonical relation $C \subset (T^*M \setminus O) \times (T^*M \setminus O)$. (Recall C is a conic Lagrangian submanifold in $(T^*M \setminus O) \times (T^*M \setminus O)$ with respect to the canonical symplectic form). We consider $A \in I^m(M, C)$ as a continuous linear operator $A : C^\infty(M) \rightarrow C^\infty(M)$ which extends continuously to $\mathcal{E}'(M) \rightarrow \mathcal{E}'(M)$ and can be represented in local charts Ω by the standard form

$$Au(x) = (2\pi)^{-n} \iint e^{i\varphi(x, y, \theta)} a(x, y, \theta) u(y) dy d\theta \quad (2.1)$$

with $x, y \in \Omega \subset \mathbb{R}^n$, $\theta \in \mathbb{R}^N \setminus O$, and $u \in C_c^\infty(\Omega)$.

In (2.1) $\varphi(x, y, \theta)$ is a nondegenerate phase function associated with C and the amplitude $a(x, y, \theta)$ is an element of the space $S^{m - \frac{1}{2}(N-n)}(\Omega \times \Omega, \mathbb{R}^N)$, both in the sense of Duistermaat [8], Hörmander [12], Taylor [19], or Trèves [20]. In this paper we are concerned only with invertible Fourier integral operators. If the inverse of $A \in I^m(M, C)$ is again to be a Fourier integral operator then the canonical relation C must be invertible, that is, C must be the graph of a diffeomorphism $\eta : T^*M \setminus O \rightarrow T^*M \setminus O$. In this case, the fact that $C = \text{graph}(\eta)$ is a conic Lagrangian submanifold of $(T^*M \setminus O) \times (T^*M \setminus O)$ is equivalent to $\eta^*\theta = \theta$; i.e. η preserves the canonical one-form θ on $T^*M \setminus O$.

Note that for this to be true it is crucial that the zero section is deleted from T^*M , otherwise η would just be a lift of a diffeomorphism $g : M \rightarrow M$.

Let $\mathcal{D}_\theta(T^*M \setminus O) = \{\eta : T^*M \setminus O \rightarrow T^*M \setminus O \mid \eta \text{ is a diffeomorphism and } \eta^*\theta = \theta\}$ be the group of homogeneous canonical diffeomorphisms of $T^*M \setminus O$. Denote the class of Fourier integral operators of order m with $C = \text{graph}(\eta)$ by $I^m(M, \eta)$. If $A_1 \in I^{m_1}(M, \eta_1)$ and $A_2 \in I^{m_2}(M, \eta_2)$ then $A_1 \circ A_2 \in I^{m_1+m_2}(M, \eta_1 \circ \eta_2)$.

The class of pseudodifferential operators of order m on M is defined as $L^m(M) = I^m(M, e)$ where e is the identity $e : T^*M \setminus O \rightarrow T^*M \setminus O$. A pseudodifferential operator P can be locally represented in the form

$$Pu(x) = (2\pi)^{-n} \iint e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) dy d\xi, \quad (2.2)$$

where $x, y \in \Omega \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $a(x, y, \xi) \in S^m(\Omega \times \Omega, \mathbb{R}^n)$. The spaces $L^{-\infty}(M) = \bigcap_{m \in \mathbb{R}} L^m(M)$ and $I^{-\infty}(M, \eta) = \bigcap_{m \in \mathbb{R}} I^m(M, \eta)$ both coincide with the space of regularizing operators on M (see e.g. Trèves [20]).

In this paper we shall consider only *formal* pseudodifferential and Fourier integral operators. These are defined as follows. Let $S^m(\Omega)$ be the subspace of $S^m(\Omega \times \Omega, \mathbb{R}^n)$ consisting of amplitudes $a(x, \xi)$ independent of y . The map associating to $p(x, \xi) \in S^m(\Omega)$ the operator $P \in L^m(\Omega)$ given by

$$Pu(x) = (2\pi)^{-n} \iint e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi \quad (2.3)$$

induces an isomorphism of $S^m(\Omega)/S^{-\infty}(\Omega)$ ($S^{-\infty}(\Omega) = \bigcap_{m \in \mathbb{R}} S^m(\Omega)$) with $L^m(\Omega)/L^{-\infty}(\Omega)$ (see e.g. Trèves [20]). Elements of $S^m(\Omega)$ are called *symbols* of order m . A *formal pseudodifferential operator* on M of order m is an element of $L^m(M)/L^{-\infty}(M)$ which has a representative in any local chart Ω of the form (2.3) with $p(x, \xi)$ defining a classical symbol, i.e. $p(x, \xi)$ has an asymptotic expansion $\sum_{j=0}^{+\infty} p_{m-j}(x, \xi)$ where each term $p_{m-j}(x, \xi)$ is smooth off $\xi=0$ and homogeneous of degree $m-j$ in ξ . We denote the space of formal pseudodifferential operators of order m by ψDO_m . The *principal symbol* $\sigma(P)(x, \xi)$ of $P \in \psi DO_m$ equals $p_m(x, \xi)$ in any local chart and is globally defined as a smooth homogeneous function of degree m on $T^*M \setminus O$.

A *formal Fourier integral operator* of order m , with canonical relation graph(η), is an element of the quotient space $I^m(M, \eta)/I^{-\infty}(M, \eta)$ which has a representative in any local chart Ω of the form (2.1) with $a(x, y, \xi)$ defining a classical symbol. We call this space $FIO_m(\eta)$ and denote

$$FIO_m = \bigcup_{\eta \in \mathcal{D}_\theta(T^*M \setminus O)} FIO_m(\eta).$$

Notice that for $-k \leq m$ we have $\psi DO_{-k-1} \subset \psi DO_m$. We let

$$\psi DO_{m,k} = \psi DO_m / \psi DO_{-k-1};$$

these are the formal pseudodifferential operators of order m , modulo those of order $-k-1$. Note that on Ω there is an isomorphism between $\psi DO_{m,k}$ and the classical symbols in $S^m(\Omega)/S^{-k-1}(\Omega)$, which in turn can be thought of as finite expansions

$\sum_{j=0}^{m+k} p_{m-j}(x, \xi)$ where $p_{m-j}(x, \xi)$ is homogeneous of degree $m-j$. Likewise we define $FIO_{m,k}(\eta) = FIO_m(\eta)/FIO_{-k-1}(\eta)$ the formal Fourier integral operators of order m with canonical relation graph(η) modulo those of order $-k-1$. In any local chart Ω we can think of an element of $FIO_{m,k}(\eta)$ as the equivalence class of an operator of the form (2.1) where $a(x, y, \theta)$ has a finite expansion $\sum_{j=1}^{m+k} a_{m_0-j}(x, y, \theta)$ with $m_0 = m - \frac{1}{2}(N-n)$. Of course, $FIO_{m,k}(e) = \psi DO_{m,k}$, where e is the identity. Finally, we let

$$FIO_{m,k} = \bigcup_{\eta \in \mathcal{D}_\theta(T^*M \setminus O)} FIO_{m,k}(\eta).$$

Notice that if $m=0$ then composition is well defined in $FIO_{0,k}$.

Let $(FIO_{0,k})_*$ and $(\psi DO_{0,k})_*$ denote the groups of invertible elements of $FIO_{0,k}$ and $\psi DO_{0,k}$ respectively. We get the following exact sequence of groups

$$I \longrightarrow (\psi DO_{0,k})_* \xrightarrow{j} (FIO_{0,k})_* \xrightarrow{\pi} \mathcal{D}_\theta(T^*M \setminus O) \longrightarrow e. \quad (2.4)$$

2.2. ILH-Lie Groups and $\mathcal{D}_\theta(T^*M \setminus O)$

A collection of groups $\{G^\infty, G^s | s \geq s_0\}$ is called an *ILH-Lie group* (inverse limit of Hilbert) if :

- (i) each G^s is a Hilbert manifold of class $C^{k(s)}$, modeled on a Hilbert space E^s , where the order of differentiability $k(s)$ tends to ∞ as $s \rightarrow \infty$;
- (ii) for each $s \geq s_0$, there are linear continuous, dense inclusions $E^{s+1} \hookrightarrow E^s$ and dense inclusions of class $C^{k(s)}$, $G^{s+1} \hookrightarrow G^s$;
- (iii) each G^s is a topological group and $G^\infty = \varprojlim G^s$ is a topological group with the inverse limit topology;
- (iv) if (U^s, φ^s) is a chart on G^s , then $(U^s \cap G^t, \varphi^s|_{U^s \cap G^t})$ is a chart on G^t , for all $t \geq s$;
- (v) group multiplication $\mu: G^\infty \times G^\infty \rightarrow G^\infty$ can be extended to a C^k -map $\mu: G^{s+k} \times G^s \rightarrow G^s$ for any s such that $k \leq k(s)$;
- (vi) inversion $v: G^\infty \rightarrow G^\infty$ can be extended to a C^k -map $v: G^{s+k} \rightarrow G^s$, for any s satisfying $k \leq k(s)$;
- (vii) right multiplication R_g by $g \in G^s$ is a $C^{k(s)}$ -map $R_g: G^s \rightarrow G^s$.

If the manifolds are Banach manifolds rather than Hilbert manifolds then $\{G^\infty, G^s | s \geq s_0\}$ is an *ILB-Lie group*.

A collection of vector spaces $\{g^\infty, g^s | s \geq s_0\}$ is called an *ILH-(ILB-)Lie algebra* if :

- (i) each g^s is a Hilbert (Banach)-space and for each $s \geq s_0$ there are linear, continuous, dense inclusions $g^{s+1} \hookrightarrow g^s$ and $g^\infty = \varprojlim g^s$ is a Fréchet space with the inverse limit topology;
- (ii) there exist bilinear, continuous, antisymmetric maps $[\cdot, \cdot]: g^{s+2} \times g^{t+2} \rightarrow g^{\min(s,t)}$ for all $s, t \geq s_0$, which satisfy the Jacobi identity on $g^{\min(s,t,r)}$ for elements in $g^{s+4} \times g^{t+4} \times g^{r+4}$.

If $\{G^\infty, G^s | s \geq s_0\}$ is an ILH-Lie group, put $g^s = T_e G^s$ and $g^\infty = \varprojlim g^s$; then $\{g^\infty, g^s | s \geq s_0\}$ is the ILH-Lie algebra of the ILH-Lie group $\{G^\infty, G^s | s \geq s_0\}$.

The classical examples of smooth ILH-Lie groups (that is each G^s is a smooth manifold, $k(s) = \infty$) are the diffeomorphism groups of a compact manifold M : $\{\mathcal{D}^\infty(M), \mathcal{D}^s(M) | s > \frac{1}{2} \dim M\}$ with ILH-Lie algebra $\{\mathcal{X}^\infty(M), \mathcal{X}^s(M) | s > \frac{1}{2} \dim M\}$; where $\mathcal{D}^s(M)$ and $\mathcal{X}^s(M)$ are the H^s -Sobolev class diffeomorphisms and vector fields on M respectively.

The group $\mathcal{D}_\theta^s(T^*M \setminus O)$ has an added difficulty in that $T^*M \setminus O$ is *not* compact, so the standard results on diffeomorphism groups as ILH-Lie groups do not apply in a straightforward manner. We shall review here, following [18] how

$$\{\mathcal{D}_\theta^\infty(T^*M \setminus O), \mathcal{D}_\theta^s(T^*M \setminus O) | s \geq \dim M + 1/2\}$$

is an ILH-Lie group with ILH-Lie algebra

$$\{\mathcal{S}^\infty(T^*M \setminus O), \mathcal{S}^{s+1}(T^*M \setminus O) | s \geq \dim M + 3/2\},$$

where

$$\mathcal{S}^s(T^*M \setminus O) = \{H: T^*M \setminus O \rightarrow \mathbb{R} | H \text{ is of class } H^s \text{ and homogeneous of degree one}\},$$

with the Poisson bracket as Lie algebra bracket. (Note the gain in one derivative at the Lie algebra level.) Let $Q = (T^*M \setminus O)/\mathbb{R}_+$ be the cosphere bundle of M and $\sigma: Q \rightarrow T^*M \setminus O$ a global section. Then $\theta_\sigma = \sigma^* \theta$ is an exact contact one form on Q .

The group of H^{s+1} -contact transformations on Q is isomorphic to the group

$$\text{Con}_\sigma^{s+1}(Q) = \{(\varphi, h) \in \mathcal{D}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R} \setminus O) \mid \varphi^* \theta_\sigma = h \theta_\sigma\}$$

for any fixed but arbitrary global section σ , where $\mathcal{D}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R} \setminus O)$ is the semidirect product of the Lie groups $\mathcal{D}^{s+1}(Q)$ and $H^{s+1}(Q, \mathbb{R} \setminus O)$ ($H^{s+1}(Q, \mathbb{R} \setminus O)$ as a multiplicative group) with composition law $(\varphi_1, h_1) \cdot (\varphi_2, h_2) = ((\varphi_1 \circ \varphi_2), h_2(h_1 \circ \varphi_2))$. $\text{Con}_\sigma^{s+1}(Q)$ is a closed Lie subgroup of the Lie group $\mathcal{D}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R} \setminus O)$. The Lie algebra of $\mathcal{D}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R} \setminus O)$ is the semidirect product $\mathcal{X}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R})$ of H^{s+1} -vector fields and H^{s+1} functions, with bracket $[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f))$. The Lie algebra of $\text{Con}_\sigma^{s+1}(Q)$ is

$$\text{con}_\sigma^{s+1}(Q) = \{(Y, g) \in \mathcal{X}^{s+1}(Q) \ltimes H^{s+1}(Q, \mathbb{R}) \mid L_Y \theta_\sigma = g \theta_\sigma\}$$

(L_Y denoting the Lie derivative along the vector field Y). The group $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$ is isomorphic (as a group) to the Lie group $\text{Con}_\sigma^{s+1}(Q)$. The isomorphism is given by $\Phi: \mathcal{D}_\theta^{s+1}(T^*M \setminus O) \rightarrow \text{Con}_\sigma^{s+1}(Q)$, $\Phi(\eta) = (\varphi, h)$ where φ is defined by $\varphi \circ \pi = \pi \circ \eta$ and h by $h \circ \pi = (f_\sigma \circ \eta)/f_\sigma$, $\sigma(\pi(\alpha_x)) = f_\sigma(\alpha_x)\alpha_x$.

The isomorphism Φ determines an ILH-Lie group structure on $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$ which is independent of σ (or independent of the Riemannian metric if σ is induced from such). Furthermore, the Lie algebra of $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$, $\mathcal{X}_\theta^{s+1}(TM \setminus O) = \{Y \in \mathcal{X}^{s+1}(T^*M \setminus O) \mid L_Y \theta = 0\}$ is isomorphic to $\mathcal{S}^{s+2}(T^*M \setminus O) = \{H \in H^{s+2}(T^*M \setminus O, \mathbb{R}) \mid H \text{ homogeneous of degree one}\}$.

2.3. A Local Section from $\mathcal{D}_\theta(T^*M \setminus O)$ into $(\text{FIO}_{0,k})_*$

We construct a local section

$$\sigma: \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O) \rightarrow (\text{FIO}_{0,k})_*$$

of the exact sequence of groups

$$I \longrightarrow (\psi D O_{0,k})_* \xrightarrow{j} (\text{FIO}_{0,k})_* \xrightarrow{\pi} \mathcal{D}_\theta(T^*M \setminus O) \longrightarrow e.$$

This is done by explicitly constructing a chart about the identity in $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$, which allows a global writing of Fourier integral operators which are close to the identity. In [1, Sect. 4], we obtained a global writing of pseudodifferential operators by constructing a global phase function for the graph of the identity $e: T^*M \setminus O \rightarrow T^*M \setminus O$. This is done as follows. Choose a linearization v on M , i.e. a smooth map $v: \Omega \subset M \times M \rightarrow TM$ (Ω an open neighborhood of the diagonal in $M \times M$) with the properties: $v(x, y) \in T_x M$, for all $(x, y) \in \Omega$, $v(x, x) = 0_x \in T_x M$ for all $x \in M$ and the tangent map of $y \mapsto v(x, y)$ is the identity map of $T_x M$ for $x = y$. For example, choosing a Riemannian metric on M and putting $v = \text{Exp}^{-1}$, where $\text{Exp} = (\tau, \text{exp}): TM \rightarrow M \times M$ is a diffeomorphism from a neighborhood U of the zero section in TM onto a neighborhood Ω of the diagonal in $M \times M$ ($\tau: TM \rightarrow M$ is the bundle projection). Then the function $\varphi_0: T^*M \times M \rightarrow \mathbb{R}$ defined by $\varphi_0(\alpha_x, y) = \alpha_x \cdot (v(x, y))$ is a smooth global phase function for graph(e) in $T^*M \times T^*M$. Now in terms of a linearization on M a global formula for pseudodifferential operators was obtained as follows. For any $x \in M$ let $\Omega_x = \{y \in M \mid (x, y) \in \Omega\}$ and $U_x = U \cap T_x M$, so that $v_x = v(x, \cdot): \Omega_x \rightarrow U_x$ is a diffeomorphism. Let $\chi(x, y)$ be a

bump function on $M \times M$ such that $\text{supp } \chi \subset \Omega$ and $\chi \equiv 1$ on a neighborhood of the diagonal. Then for any classical symbol $a(x, \xi) \in S^m(\Omega)$,

$$Pu(x) \equiv (2\pi)^{-n} \int_{T_x^*M} a(x, \xi) d\xi \int_{\Omega_x} \chi(x, y) e^{i\varphi_0(\alpha_x, y)} u(y) |\det_{(dx, dy)} v_x| dy \quad (2.5)$$

defines a classical pseudodifferential operator P of order m on M .

Now by perturbing the global phase function φ_0 of $\text{graph}(e)$ by “small” functions $H \in \mathcal{S}^{s+2}(T^*M \setminus O)$ we obtain global phase functions for Lagrangian submanifolds close to $\text{graph}(e)$ in $(T^*M \setminus O) \times (T^*M \setminus O)$ and with these a global writing of Fourier integral operators close to the identity. Let $H \in \mathcal{S}^{s+2}(T^*M \setminus O)$ be close to zero and define $\varphi_H : (T^*M \setminus O) \times M \rightarrow \mathbb{R}$ by

$$\varphi_H(\alpha_x, y) = \varphi_0(\alpha_x, y) + H(\alpha_x).$$

Then there exists an $\eta \in \mathcal{D}_\theta^{s+1}(T^*M \setminus O)$ such that φ_H is a global phase function for $\text{graph}(\eta)$. This defines a map $H \mapsto \eta$, which is a bijection from a neighborhood \mathcal{V} of zero in $\mathcal{S}^{s+2}(T^*M \setminus O)$ onto a neighborhood \mathcal{U} of the identity in $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$. Moreover $(\mathcal{U}, \Phi, \mathcal{S}^{s+2}(T^*M \setminus O))$ is a smooth chart at the identity of $\mathcal{D}_\theta^{s+1}(T^*M \setminus O)$, where $\Phi : \mathcal{U} \subset \mathcal{D}_\theta^{s+1}(T^*M \setminus O) \rightarrow \mathcal{S}^{s+2}(T^*M \setminus O)$ is defined by

$$\Phi(\eta)(\alpha_x) = -\alpha_x \cdot v(x, \tau^* \eta^{-1}(\alpha_x))$$

for all $\alpha_x \in T^*M \setminus O$.

Now for conic Lagrangian submanifolds of $T^*M \times T^*M$ which are close to the diagonal, or equivalently for canonical transformations $\eta \in \mathcal{D}_\theta(T^*M \setminus O)$ which are close to the identity we have global phase functions. Therefore we can globally write a representative of a classical Fourier integral operator $A \in \text{FIO}_m(\eta)$ if $\eta \in \mathcal{U} \cap \mathcal{D}_\theta(T^*M \setminus O)$ as follows: if $\chi(x, y)$ is a bump function as above and $a(x, \xi)$ is a representative of a classical symbol of order m , then

$$Au(x) = (2\pi)^{-n} \int_{T_x^*M} a(x, \xi) d\xi \int_{\Omega_x} \chi(x, y) e^{i\varphi_H(\alpha_x, y)} u(y) |\det_{(dx, dy)} v_x| dy. \quad (2.6)$$

Now we can define a local section

$$\sigma : \mathcal{U} \cap \mathcal{D}_\theta(T^*M \setminus O) \rightarrow (\text{FIO}_{0,k})_*$$

of the sequence (2.4) as follows:

$$\sigma(\eta)u(x) = (2\pi)^{-n} \int_{T_x^*M} d\xi \int_{\Omega_x} \chi(x, y) e^{i\varphi_H(\alpha_x, y)} u(y) |\det_{(dx, dy)} v_x| dy \quad (2.7)$$

for $\eta \in \mathcal{U} \cap \mathcal{D}_\theta(T^*M \setminus O)$, where $\varphi_H(\alpha_x, y) = \varphi_0(\alpha_x, y) + H(\alpha_x)$ and $H = \Phi(\eta)$, so $\varphi_H(\alpha_x, y) = \alpha_x \cdot (v(x, y)) - \alpha_x \cdot v(x, \tau^* \eta^{-1}(\alpha_x))$. Explicitly, we have

$$\sigma(\eta)u(x) = (2\pi)^{-n} \int_{T_x^*M} d\xi \int_{\Omega_x} \chi(x, y) e^{i\alpha_x \cdot (v(x, y) - v(x, \tau^* \eta^{-1}(\alpha_x)))} u(y) \left| \det \frac{\partial v(x, y)}{\partial y} \right| dy.$$

2.4. The Lie Group Structure of $(\psi DO_{0,k})_*$

The space $(\psi DO_0)_*$ was given the structure of a topological group in [1] as follows. Let $\varphi_0 : T^*M \times M \rightarrow \mathbb{R}$ be a global phase function for $e \in \mathcal{D}_\theta(T^*M \setminus O)$; we can write a representative of $[P] \in \psi DO_m$ in the form (2.5) where $a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi)$,

$a_{m-j}(x, \xi)$ being a smooth function on $T^*M \setminus O$, homogeneous of degree $m-j$. A section $\sigma: Q \rightarrow T^*M \setminus O$ identifies ψDO_m with the infinite product $C^\infty(Q) \times C^\infty(Q) \times \dots$ by $[P] \mapsto (a_m \circ \sigma, a_{m-1} \circ \sigma, \dots)$. We endow ψDO_m with the topology induced by this identification. Then $\psi DO_{m,k}$ is isomorphic to the $(m+k+1)$ -fold product $C^\infty(Q) \times \dots \times C^\infty(Q)$ by extending the j th function to $T^*M \setminus O$ by homogeneity of degree $m-j$, $0 \leq j \leq m+k$, and then using φ_0 to give an identification of $\psi DO_{m,k}$ with the finite sequences $(a_m(x, \xi), a_{m-1}(x, \xi), \dots, a_{m-k}(x, \xi))$.

We use H^s structures on the symbol spaces to define $\psi DO_{m,k}^s$. Let $a_l(x, \xi)$ be a smooth function on $T^*M \setminus O$ which is homogeneous of degree l in ξ . Define (for $s > \dim M$)

$$\|a_l(x, \xi)\|_s = \|a_l \circ \sigma\|_s, \quad (2.8)$$

where the norm $\|\cdot\|_s$ on the right is the H^s norm of $a_l \circ \sigma$ as a function on the compact manifold Q . Let $\mathcal{S}_l^s(T^*M \setminus O)$ be the completion of the set of smooth functions on $T^*M \setminus O$, homogeneous of degree l , with respect to this topology.

Define the s -norm of an element $[P] \in \psi DO_{m,k}$ as follows. If $\sum_{j=0}^{m+k} a_{m-j}$ is the symbol of P , put

$$\|P\|_{m,k;s}^2 = \|a_m\|_{s+m}^2 + \|a_{m-1}\|_{s+m-1}^2 + \dots + \|a_{-k}\|_s^2, \quad (2.9)$$

where $\|a_{m-j}\|_{s+m+k-j}$ is the norm of a_{m-j} in $\mathcal{S}_{m-j}^{s+k+m-j}(T^*M \setminus O)$ given by some choice of a section $\sigma: Q \rightarrow T^*M \setminus O$. Let $\psi DO_{m,k}^s$ denote the completion of $\psi DO_{m,k}$ with respect to this norm. This makes $\psi DO_{m,k}^s$ a Hilbert space. If $m=0$ composition in $\psi DO_{0,k}$ is well defined and extends continuously to $\psi DO_{0,k}^s$.

Thus $\psi DO_{0,k}^s$ is a Hilbert algebra. It follows that the subset $(\psi DO_{0,k}^s)_*$ of invertible elements in $\psi DO_{0,k}^s$ is open and that inversion, $P \mapsto P^{-1}$, is a homeomorphism of $(\psi DO_{0,k}^s)_*$. Therefore $(\psi DO_{0,k}^s)_*$ is a Hilbert Lie group. Summarizing we have the following result of [1]:

$\{(\psi DO_{0,k}^s)_, (\psi DO_{0,k}^s)_* | s > \dim M\}$ is an ILH-Lie group where each $(\psi DO_{0,k}^s)_*$ is a smooth Hilbert Lie group. Its ILH-Lie algebra is $\{\psi DO_{0,k}, \psi DO_{0,k}^s | s > \dim M\}$.*

3. Exact Sequences of ILH-Lie Groups

We study here an exact sequence

$$I \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow e \quad (3.1)$$

of groups where \mathcal{H} and \mathcal{Q} have ILH-Lie group structures. We desire an apparatus for inducing an ILH-Lie group structure on \mathcal{G} .

We begin with a simple result on topological groups

Proposition 3.1. *Let*

$$I \longrightarrow H \xrightarrow{j} G \xrightarrow{\pi} Q \longrightarrow e \quad (3.2)$$

be an exact sequence of groups. Assume that H and Q are topological groups. Let $U \subset Q$ be a neighborhood of the identity, e , and let $\sigma: U \rightarrow G$ be a local section. Let $V \subset U$ be a neighborhood of e in Q such that $V \cdot V^{-1} \subset U$ and assume

A) the map $V \times V \times H \rightarrow H$ given by

$$(\eta_1, \eta_2, h) \mapsto \sigma(\eta_1)\sigma(\eta_2)^{-1}h\sigma(\eta_1\eta_2^{-1})^{-1}$$

is continuous;

B) for each $g \in G$ and $W \subset U$ such that $\pi(g)W\pi(g)^{-1} \subset U$ the map $W \times H \rightarrow H$ given by

$$(\eta, h) \mapsto gh\sigma(\eta)g^{-1}\sigma(\pi(g)\eta\pi(g)^{-1})^{-1}$$

is continuous.

Then G can be made a topological group so that j , π , and σ are continuous and π is open.

Proof. Give G a topology as follows. Let $\psi: \pi^{-1}(U) \rightarrow U \times H$ be given by

$$g \mapsto (\pi(g), g\sigma(\pi(g))^{-1}).$$

ψ is a bijection with inverse $(\eta, h) \mapsto h\sigma(\eta)$ so we can give $\pi^{-1}(U)$ a topology by declaring ψ to be a homeomorphism. Now for a basis of the topology on G move the open sets in $\pi^{-1}(U)$ around by right translation. This, of course, makes right translation continuous. It is also clear that j , π , and σ are continuous and that π is open.

To see that G is a topological group we must check that $(x, y) \mapsto xy^{-1}$ is continuous. For $a, b \in G$ fixed, let W be an open set in U such that $\pi(ab^{-1})W\pi(ab^{-1})^{-1} \subset V$. We will show that $(x, y) \mapsto xy^{-1}$ is continuous on the open set

$$\pi^{-1}(W)a \times \pi^{-1}(W)b \subset G \times G.$$

Thus, we consider $x = ua$ and $y = vb$ for $u, v \in \pi^{-1}(W)$ and we want to show

$$(u, v) \mapsto uab^{-1}v^{-1}$$

is continuous on $\pi^{-1}(W) \times \pi^{-1}(W)$. But

$$\begin{aligned} uab^{-1}v^{-1} &= u(ab^{-1}vba^{-1})^{-1}ab^{-1} \\ &= R_{ab^{-1}}(u(I_{ab^{-1}}v)^{-1}), \end{aligned}$$

where $R_{ab^{-1}}$ is right multiplication by ab^{-1} and $I_{ab^{-1}}$ is conjugation by ab^{-1} . We have already mentioned that $R_{ab^{-1}}$ is continuous so it suffices to show that the maps

$$(i) \quad \pi^{-1}(V) \times \pi^{-1}(V) \rightarrow \pi^{-1}(U)$$

$$(x, y) \mapsto xy^{-1};$$

$$(ii) \quad I_{ab^{-1}}: \pi^{-1}(W) \rightarrow \pi^{-1}(V)$$

$$x \mapsto ab^{-1}xba^{-1}$$

are continuous.

Using the homeomorphism ψ , we can write the first map as

$$(V \times H) \times (V \times H) \rightarrow U \times H$$

$$((\eta_1, h_1), (\eta_2, h_2)) \mapsto (\eta_1\eta_2^{-1}, h_1\sigma(\eta_1)\sigma(\eta_2)^{-1}h_2^{-1}\sigma(\eta_1\eta_2^{-1})^{-1}).$$

This map is continuous by A) and the fact that H and Q are topological groups. In the same manner, the map (ii) becomes

$$W \times H \rightarrow U \times H$$

$$(\eta, h) \rightarrow (\pi(ab^{-1})\eta\pi(ab^{-1})^{-1}, ab^{-1}h\sigma(\eta)ba^{-1}\sigma(\pi(ab^{-1})\eta\pi(ab^{-1})^{-1}))$$

which is continuous by B). \square

Remark. If Q is connected it is enough to check A). Indeed A) gives that (i) is continuous and thus I_g is continuous for g in a small neighborhood V_0 of I . Then a general g can be written $g = g_1 g_2 \dots g_k$ with the g_i 's in a small neighborhood of I (because $\pi(g) = q = q_1 \dots q_k$ with q_i near e) so (ii) must be continuous in general.

We now return our attention to the exact sequence (3.1). We assume \mathcal{H} and \mathcal{Q} are topological groups with ILH structures $\{H^s, s \geq s_0\}$ and $\{Q^t, t \geq t_0\}$. We want to make \mathcal{G} an ILH-Lie group.

Let \mathcal{H}^s and \mathcal{Q}^t be the spaces \mathcal{H} and \mathcal{Q} with the coarser topologies of H^s and Q^t . \mathcal{H}^s and \mathcal{Q}^t are topological groups since they are subgroups of H^s and Q^t . Suppose there is an open set $U \subset \mathcal{Q}^{t_0}$ (and thus U is open in $\mathcal{Q}^t \forall t \geq t_0$) and $\sigma: U \rightarrow \mathcal{G}$, a local section. Let $V \subset U$, open in \mathcal{Q}^{t_0} , be such that $V \cdot V^{-1} \subset U$ and let V^t denote V with the topology induced from \mathcal{Q}^t . Assume that for each $t \geq t_0$ there is an $s(t)$ such that

A') the map $V^t \times V^t \times \mathcal{H}^{s(t)} \rightarrow \mathcal{H}^{s(t)}$ defined by

$$(\eta_1, \eta_2, h) \mapsto \sigma(\eta_1) \times \sigma(\eta_2)^{-1} h \sigma(\eta_1 \eta_2^{-1})^{-1}$$

is continuous;

B') for each $g \in \mathcal{G}$ and $W \subset U$ with $\pi(g)W\pi(g)^{-1} \subset U$, the map $W^t \times \mathcal{H}^{s(t)} \rightarrow \mathcal{H}^{s(t)}$ defined by

$$(\eta, h) \mapsto gh\sigma(\eta)g^{-1}\sigma(\pi(g)\eta\pi(g)^{-1})^{-1} \quad \text{is continuous.}$$

Then, applying Proposition 3.1 for each t , we make \mathcal{G} into a topological group \mathcal{G}^t , so that

$$I \rightarrow \mathcal{H}^{s(t)} \rightarrow \mathcal{G}^t \rightarrow \mathcal{Q}^t \rightarrow e \quad (3.3)$$

is an exact sequence of topological groups $\forall t \geq t_0$.

Giving \mathcal{G} the inverse limit topology it is also a topological group, and (3.1) becomes an exact sequence of topological groups.

The next step is to "complete" the groups in (3.3). In order to do this we need to say what a Cauchy sequence is, i.e. we need a uniform structure. We use the *right uniform structure* of a topological group (see e.g. Bourbaki [6]). Namely, a sequence $\{x_n\}$ in \mathcal{G}^t converges to x in \mathcal{G}^t if given a neighborhood V of x in \mathcal{G}^t , there is an N such that $xx_n^{-1} \in V$ for all $n > N$. Similarly, $\{x_n\}$ is Cauchy in \mathcal{G}^t if for any neighborhood V of the identity I in \mathcal{G}^t there is an N such that $x_n x_m^{-1} \in V$ for all $m, n > N$.

Lemma 3.2. *Let X be a locally Hilbert topological group. Then X is complete in its right uniform structure.*

Proof. To show X is complete it is enough to show that a neighborhood of the identity is complete. Since X is locally Hilbert, there is a neighborhood V of the identity, e , in X which is homeomorphic to the closed unit ball in Hilbert space. Let

$\{x_n\}$ be a Cauchy sequence in V , then given $\varepsilon > 0$ there is an $N > 0$ s.t.

$$\|x_m x_n^{-1} - e\| < \varepsilon \quad \text{for all } n, m > N$$

(here $\|\cdot\|$ denotes the induced norm in V).

But $\|x_n\| < 1$ so we must have

$$\|x_m - x_n\| < \varepsilon \quad \text{for } n, m > N.$$

Hence $\{x_n\}$ is Cauchy in the closed unit ball of Hilbert space, which is clearly complete. \square

Applying Lemma 3.2 to H^s and noting that \mathcal{H}^s is dense in H^s we see that the completion of \mathcal{H}^s must be H^s . Similarly the completion of \mathcal{Q}^t is Q^t .

Let G^t denote the completion of \mathcal{G}^t with respect to its right uniform structure. At the moment, we don't know that G^t is a group, however, it is not hard to see that multiplication extends to a continuous map

$$G^t \times G^t \rightarrow G^t$$

since it maps Cauchy sequences to Cauchy sequences. Thus G^t is a topological semigroup. It is also clear that j and π map Cauchy sequences to Cauchy sequences so they extend to continuous maps, giving the exact sequence

$$I \longrightarrow H^{s(t)} \xrightarrow{j} G^t \xrightarrow{\pi} Q^t \longrightarrow e \quad (3.4)$$

of topological semigroups. Since $H^{s(t)}$ and Q^t are groups it follows that G^t is a group. Indeed, if $g \in G^t$ we must produce an inverse for g . Let $W = \pi^{-1}(\pi(g)^{-1})$, then by the exactness of (3.4), $gW = H$. Thus there is a $w \in W$ s.t. $gw = e$, i.e. w is g^{-1} . It does not in general follow that inversion is continuous. However we have the following

Lemma 3.3. *Let $\tilde{U}^{t_0} \subset Q^{t_0}$ be an open set with $\tilde{U}^{t_0} \cap \mathcal{Q}^{t_0} = U^{t_0}$. For $t \geq t_0$ let $\tilde{U}^t = \tilde{U}^{t_0} \cap Q^t$. Similarly let $\tilde{V}^{t_0} \subset \tilde{U}^{t_0}$ be open with $\tilde{V}^{t_0} \cap \mathcal{Q}^{t_0} = V^{t_0}$ and set $\tilde{V}^t = \tilde{V}^{t_0} \cap Q^t$, $t \geq t_0$. Suppose that $\sigma: U \rightarrow \mathcal{G}$ extends to a continuous local section $\tilde{\sigma}: \tilde{U}^t \rightarrow G^t$. Assume*

A'') the map $V^t \times V^t \times \mathcal{H}^{s(t)} \rightarrow \mathcal{H}^{s(t)}$ from assumption A') actually extends by $\tilde{\sigma}$ to a continuous map $\tilde{V}^t \times \tilde{V}^t \times H^{s(t)} \rightarrow H^{s(t)}$.

Then G^t is a topological group.

Proof. Notice that for $g \in \mathcal{G}$, the map $I_g: \mathcal{G} \rightarrow \mathcal{G}$ is uniformly continuous so it extends to a continuous map

$$I_g^t: G^t \rightarrow G^t.$$

A quick glance at the proof of Proposition 3.1 shows that its conclusion holds also if I_g^t is continuous only for g in a dense subset of G^t . This observation, along with A'') allows us to apply Proposition 3.1 to give G^t a topology making (3.4) an exact sequence of topological groups. Since $\mathcal{G}^t \subset G^t$ is dense, this topology on G^t must be the same as that given by completing \mathcal{G}^t . \square

The next step is to give G^t a manifold structure. To do this we declare $\psi_I: \pi^{-1}(U^t) \rightarrow U^t \times H^{s(t)}$ defined by

$$g \mapsto (\pi(g), g\sigma(\pi(g))^{-1}) \quad (3.5)$$

to be a principal bundle chart. We move this around on G^t by right translation as follows. Let $g_0 \in G^t$ and $q_0 = \pi(g_0)$. On $\pi^{-1}(U^t q_0)$ define

$$\psi_{g_0} : \pi^{-1}(U^t q_0) \rightarrow U^t q_0 \times H^{s(t)} \quad \text{by} \quad g \mapsto (\pi(g), g g_0^{-1} \sigma(\pi(g) q_0^{-1})^{-1}). \quad (3.6)$$

In order that the charts $(\pi^{-1}(U^t q_0), \psi_{g_0})$ define an atlas we need an overlap condition, namely, we need to check that for $g_0, \tilde{g}_0 \in G^t$, the map

$$\psi_{\tilde{g}_0} \circ \psi_{g_0}^{-1} : (U^t q_0 \cap U^t \tilde{q}_0) \times H^{s(t)} \rightarrow (U^t q_0 \cap U^t \tilde{q}_0) \times H^{s(t)}$$

is $C^{k(t)}$. This map is

$$(\eta, h) \mapsto (\eta, h \sigma(\eta q_0^{-1}) g_0 \tilde{g}_0^{-1} \sigma(\eta \tilde{q}_0^{-1})^{-1}).$$

So, in order to guarantee that G^t is a $C^{k(t)}$ Hilbert manifold we add to our list of assumptions the following additional condition:

C) for $g_0, \tilde{g}_0 \in G^t$ let $\pi(g_0) = q_0$, $\pi(\tilde{g}_0) = \tilde{q}_0$ and assume that the map

$$(U^t q_0) \cap (U^t \tilde{q}_0) \times H^{s(t)} \rightarrow H^{s(t)}$$

given by

$$(\eta, h) \mapsto h \sigma(\eta q_0^{-1}) g_0 \tilde{g}_0^{-1} \sigma(\eta \tilde{q}_0^{-1})^{-1}$$

is $C^{k(t)}$, where $k(t)$ is an increasing function of t .

Remarks. 1) The inclusion $G^{t+1} \hookrightarrow G^t$ is $C^{k(t)}$ since in coordinates this is the inclusion

$$U_{q_0}^{t+1} \times H^{s(t+1)} \rightarrow U_{q_0}^t \times H^{s(t)}.$$

2) It is also easy to see that condition (iv) in the definition of ILH-Lie groups is satisfied since it holds for Q^t and $H^{s(t)}$.

3) Since we define the manifold structure by right translation, it is automatic that right translation, R_g , is $C^{k(t)}$ for $g \in G^t$ fixed. Indeed, in the charts $(\pi^{-1}(U^t)g_0, \psi_{g_0})$ and $(\pi^{-1}(U^t)g_0 g, \psi_{g_0 g})$ R_g is just the map

$$(\eta, h) \mapsto (\eta \pi(g), h).$$

To finish the construction of an ILH-Lie group structure for \mathcal{G} we need a condition so that multiplication, $M : G^{r+k} \times G^r \rightarrow G^r$ is C^k for $k \leq k(r)$.

For $a \in G^{r+k}$ and $b \in G^r$ set $\alpha = \pi(a)$, $\beta = \pi(b)$. We have charts $(\pi^{-1}(U^{r+k}\alpha), \psi_a)$, $(\pi^{-1}(U^r\beta), \psi_b)$, and $(\pi^{-1}(U^r\alpha\beta), \psi_{ab})$ around a , b , and ab . In these charts M is the map

$$(U^{r+k}\alpha \times H^{s(r+k)}) \times (U^r\beta \times H^{s(r)}) \rightarrow U^r\alpha\beta \times H^{s(r)}$$

given by

$$((\eta_1, h_1), (\eta_2, h_2)) \mapsto (\eta_1 \eta_2, h_1 \sigma(\eta_1 \alpha^{-1}) a h_2 \sigma(\eta_2 \beta^{-1}) a^{-1} \sigma(\eta_1 \eta_2 \beta^{-1} \alpha^{-1})^{-1}).$$

So we need to assume

D) for $a \in G^{r+k}$, $b \in G^r$, $\alpha = \pi(a)$, $\beta = \pi(b)$ the map

$$U^{r+k}\alpha \times U^r\beta \times H^{s(r)} \rightarrow H^{s(r)}$$

given by

$$(\eta_1, \eta_2, h) \mapsto \sigma(\eta_1 \alpha^{-1}) a h \sigma(\eta_2 \beta^{-1}) a^{-1} \sigma(\eta_1 \eta_2 \beta^{-1} \alpha^{-1})^{-1}$$

is C^k , as long as $k \leq k(r)$.

In summary, we have the following

Theorem 3.4. *Let*

$$I \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow e \quad (*)$$

be an exact sequence of groups where \mathcal{H} and \mathcal{Q} have ILH structures. Suppose a local section $\sigma : U \rightarrow \mathcal{G}$ exists, satisfying A''), B'), C), and D). Then \mathcal{G} has an ILH-Lie group structure making () an exact sequence of ILH-Lie groups. If \mathcal{Q} is connected B') follows from A'').*

4. A Lie Group Structure for $(\text{FIO}_{0,k})_*$

We are now ready to combine the previous sections to define an ILH-Lie group structure for $(\text{FIO}_{0,k})_*$. Using the local section defined in Sect. 2 and the general theorems about ILH groups in Sect. 3 we first define a topological Lie group structure on $(\text{FIO}_{0,k})_*$ and then, using the ILH-Lie group structures of $\mathcal{D}_\theta(T^*M \setminus O)$ and of $(\psi DO_{0,k})_*$ we will obtain an ILH-Lie group structure for $(\text{FIO}_{0,k})_*$.

In the exact sequence (3.1) we have for the ILH-Lie groups $\mathcal{H} = \{H^s, s \geq s_0\}$ and $\mathcal{Q} = \{Q^t, t \geq t_0\}$, $\mathcal{H} = (\psi DO_{0,k})_*$, $H^s = (\psi DO_{0,k}^s)_*$, $s_0 = 1 + \frac{1}{2} \dim T^*M$, $\mathcal{Q} = \mathcal{D}_\theta(T^*M \setminus O)$, $Q^t = \mathcal{D}_\theta^t(T^*M \setminus O)$, $t_0 = 1 + \frac{1}{2} \dim T^*M$. Furthermore, for the group \mathcal{G} we have $\mathcal{G} = (\text{FIO}_{0,k})_*$. For technical reasons that will become apparent we restrict ourselves to the identity components in $(\text{FIO}_{0,k})_*$ and $\mathcal{D}_\theta(T^*M \setminus O)$, but we keep for them the same notation. So we have the exact sequence of groups

$$I \longrightarrow (\psi DO_{0,k})_* \xrightarrow{j} (\text{FIO}_{0,k})_* \xrightarrow{\pi} \mathcal{D}_\theta(T^*M \setminus O) \longrightarrow e \quad (4.1)$$

Moreover we have the local section

$$\sigma : \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O) \rightarrow (\text{FIO}_{0,k})_* \quad (4.2)$$

defined by (2.7).

To conclude that $(\text{FIO}_{0,k})_*$ is a topological group such that the maps j , π , and σ are continuous, and π is open, we must verify the conditions A) and B) of Proposition 3.1. In accordance with the remark following Proposition 3.1 we need only check condition A) as long as we restrict our attention to the FIO's sitting over the identity component $\mathcal{D}_\theta(T^*M \setminus O)_0$ of $\mathcal{D}_\theta(T^*M \setminus O)$. From now on $(\text{FIO}_{0,k})_*$ will denote $\pi^{-1}(\mathcal{D}_\theta(T^*M \setminus O)_0)$.

Proposition 4.1. *Let $\sigma : \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O) \rightarrow (\text{FIO}_{0,k})_*$ be the local section defined by (2.7), and let $V \subset \mathcal{U}$ be a neighborhood of e in $\mathcal{D}_\theta(T^*M \setminus O)$ such that $V \cdot V^{-1} \subset \mathcal{U}$. Then condition (A) is satisfied.*

(A) *The map*

$$\begin{aligned} V \times V \times (\psi DO_{0,k})_* &\rightarrow (\psi DO_{0,k})_* \\ (\eta_1, \eta_2, P) &\mapsto \sigma(\eta_1) \sigma(\eta_2)^{-1} P \sigma(\eta_1 \eta_2^{-1})^{-1} \end{aligned}$$

is continuous.

Proof. We need to see that the symbol of the pseudodifferential operator $\sigma(\eta_1)\sigma(\eta_2)^{-1}P\sigma(\eta_1\eta_2^{-1})^{-1}$ depends continuously on η_1, η_2 , and P . To do this it is enough to see that if A and B are Fourier integral operators near the identity then the symbols of AB and A^{-1} depend continuously on the symbols and phase functions of A and B . Here the term “symbol” has to be interpreted as being given by the fixed phase functions discussed in Sect. 2. We make this statement more precise in the following two lemmas. The proofs are fairly technical but essentially follow from the stationary phase arguments in Sect. 3.2 of Hörmander [12] along with some slightly more precise computations found in Chap. 10 of Kumano-go [13].

For the lemmas we work in a local coordinate patch Ω on M so that the phase function φ_0 of graph (e) has the form $\varphi_0(x, \xi; y) = (x - y) \cdot \xi$. We let $S^{0,k;s}(\Omega \times \mathbb{R}^n)$ denote the space of classical symbols of order zero modulo those of order $-k-1$, endowed with the Hilbert space structure of $\psi DO_{0,k}^s(\Omega)$, given by the identification

$$\begin{aligned} S^{0,k;s}(\Omega \times \mathbb{R}^n) &\rightarrow \psi DO_{0,k}^s(\Omega) \\ a(x, \xi) &\mapsto P_a, \end{aligned}$$

where

$$P_a u(x) = (2\pi)^{-n} \iint e^{i(x-y) \cdot \xi} a(x, \xi) u(y) dy d\xi.$$

$S^{0,k}(\Omega \times \mathbb{R}^n)$ denotes the inverse limit of $S^{0,k;s}(\Omega \times \mathbb{R}^n)$. Then a Fourier integral operator $A \in \pi^{-1}(\mathcal{U})$ can be written in the form

$$Au(x) = (2\pi)^{-n} \iint e^{i((x-y) \cdot \xi + H(x, \xi))} a(x, \xi) u(y) dy d\xi, \quad (4.3)$$

where $a(x, \xi) \in S^{0,k}(\Omega \times \mathbb{R}^n)$ and $H(x, \xi) = \Phi(\pi(A)) \in \mathcal{S}(\mathbb{R}^n \setminus O)$; $\Phi: \mathcal{U} \rightarrow \mathcal{S}(\mathbb{R}^n \setminus O)$ being the map described in Sect. 2.

Lemma 4.2. *Let $W \subset \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O)$ be an open set with $W \circ W \subset \mathcal{U}$ and let \mathcal{W} denote $\Phi(W) \subset \mathcal{S}(\mathbb{R}^n \setminus O)$. For $\eta_1, \eta_2 \in W$ let $H_1 = \Phi(\eta_1)$, $H_2 = \Phi(\eta_2)$, and $H = \Phi(\eta_1 \circ \eta_2)$. Let A and B be the Fourier integral operators given by*

$$\begin{aligned} Au(x) &= (2\pi)^{-n} \iint e^{i((x-y) \cdot \xi + H_1(x, \xi))} a(x, \xi) u(y) dy d\xi \\ Bu(x) &= (2\pi)^{-n} \iint e^{i((x-y) \cdot \xi + H_2(x, \xi))} b(x, \xi) u(y) dy d\xi \end{aligned}$$

For $a(x, \xi), b(x, \xi) \in S^{0,k}(\Omega \times \mathbb{R}^n)$. Define the symbol $c(x, \xi) \in S^{0,k}(\Omega \times \mathbb{R}^n)$ by

$$A \circ Bu(x) = (2\pi)^{-n} \iint e^{i((x-y) \cdot \xi + H(x, \xi))} c(x, \xi) u(y) dy d\xi$$

then the map

$$S^{0,k} \times S^{0,k} \times \mathcal{W} \times \mathcal{W} \rightarrow S^{0,k}$$

given by

$$(a(x, \xi), b(x, \xi), H_1, H_2) \mapsto c(x, \xi)$$

is continuous. This map extends to a C^r map

$$S^{0,k;\tilde{s}(t)} \times S^{0,k;\tilde{s}(t)} \times \mathcal{W}^{-t+r} \times \mathcal{W}^{-t+r} \rightarrow S^{0,k;s(t)},$$

where $\tilde{s}(t) = 2(t - k - 1)$.

Proof.

$$\begin{aligned} Bu(y) &= (2\pi)^{-n} \int e^{i(y \cdot \xi + H_2(y, \xi))} b(y, \xi) \hat{u}(\xi) d\xi \\ A \circ Bu(x) &= (2\pi)^{-n} \int \int e^{i[(x-y) \cdot \eta + H_1(x, \eta) + y \cdot \xi + H_2(y, \xi)]} \\ &\quad \times a(x, \eta) b(y, \xi) \hat{u}(\xi) d\xi dy d\eta \\ &= (2\pi)^{-n} \int \int e^{i(x \cdot \xi + H(x, \xi))} c(x, \xi) \hat{u}(\xi) d\xi \end{aligned}$$

so

$$c(x, \xi) \sim (2\pi)^{-n} \int \int e^{i\varphi(x, y, \xi, \eta)} a(x, \eta) b(y, \xi) dy d\eta \quad (4.4)$$

with

$$\varphi(x, y, \xi, \eta) = (x - y) \cdot (\eta - \xi) + H_1(x, \eta) + H_2(y, \xi) - H(x, \xi).$$

We will use the method of stationary phase to evaluate (4.4). First we need to “cut off the edges.” There exist positive constants C_1, C_2 , so that

$$C_1|\xi| < |\eta| < C_2|\xi| \quad \text{when} \quad \xi - \eta + d_* H_2(y, \xi) = 0.$$

Choose a smooth function $\chi(\eta, \xi)$, homogeneous of degree zero, whose support lies in the cone where $C_1|\xi|/3 < |\eta| < 3C_2|\xi|$ and which has the value 1 on the cone $C_1|\xi|/2 < |\eta| < 2C_2|\xi|$. Set $\alpha(x, y, \xi, \eta) = \chi(\eta, \xi) a(x, \eta) b(y, \xi)$ and $r(x, y, \xi, \eta) = (1 - \chi(\eta, \xi)) a(x, \eta) b(y, \xi)$. Then $\alpha(x, y, \xi, \eta) \in S^{0, k}(\Omega \times \mathbb{R}^{3n})$ and $\int \int e^{i\varphi(x, y, \xi, \eta)} r(x, y, \xi, \eta) dy d\eta \in S^{-\infty}$ so to compute $c(x, \xi)$ asymptotically, it is enough to consider

$$(2\pi)^{-n} \int \int e^{i\varphi(x, y, \xi, \eta)} \alpha(x, y, \xi, \eta) dy d\eta.$$

Let $X(x, \xi)$ and $\Xi(x, \xi)$ be defined by

$$\begin{aligned} X(x, \xi) &= x + d_2 H_1(x, \Xi(x, \xi)) \\ \Xi(x, \xi) &= \xi + d_1 H_2(X(x, \xi), \xi). \end{aligned} \quad (4.5)$$

Then from [13, Sect. 10.5], we have that

$$\begin{aligned} H(x, \xi) &= H_1(x, \Xi(x, \xi)) + H_2(X(x, \xi), \xi) + x \cdot \Xi(x, \xi) \\ &\quad + X(x, \xi) \cdot \xi - x \cdot \xi - X(x, \xi) \cdot \Xi(x, \xi). \end{aligned}$$

Define new variable (z, ζ) by

$$\begin{aligned} z &= (1 + |\xi|^2)^{1/4} (y - X(x, \xi)) \\ \zeta &= (1 + |\xi|^2)^{-1/4} (\eta - \Xi(x, \xi)) \end{aligned} \quad (4.6)$$

so

$$\begin{aligned} \varphi(z, \zeta; x, \xi) &= -z \cdot \zeta + \{H_1(x, \Xi + (1 + |\xi|^2)^{1/4} \zeta) - H_1(x, \Xi) + (1 + |\xi|^2)^{1/4} (x \cdot \zeta - X \cdot \zeta)\} \\ &\quad + \{H_2(X + (1 + |\xi|^2)^{-1/4} z, \xi) - H_2(X, \xi) + (1 + |\xi|^2)^{-1/4} (z \cdot \xi - z \cdot \Xi)\}. \end{aligned}$$

A Taylor expansion in (z, ζ) now gives

$$\begin{aligned} \varphi &= -\theta(z, \zeta; x, \xi) \\ &\quad + \sum_{3 \leq |\alpha| \leq N-1} \frac{1}{\alpha!} \left[\left(\frac{\partial}{\partial \xi} \right)^\alpha H_1(x, \Xi) (1 + |\xi|^2)^{1/4} \zeta^\alpha + \left(\frac{\partial}{\partial x} \right)^\alpha H_2(X, \xi) (1 + |\xi|^2)^{-1/4} z^\alpha \right] \\ &\quad + R_N, \end{aligned}$$

where $R_N \in S^{-KN}$ for some $K > 0$ and

$$\theta = z \cdot \zeta - \frac{1}{2} \{ (1 + |\xi|^2)^{1/2} d_2^2 H_1(x, \Xi) \zeta \cdot \zeta + (1 + |\xi|^2)^{-1/2} d_1^2 H_2(X, \xi) z \cdot z \}.$$

Thus

$$\begin{aligned} c(x, \xi) \sim \int \int e^{-\theta(z, \zeta; x, \xi)} \alpha(z, \zeta; x, \xi) & \left[1 + \sum_{k=1}^{\infty} \frac{1}{k!} \right. \\ & \times \left\{ \sum_{|\alpha| \geq 3} \frac{1}{\alpha!} \left[\left(\frac{\partial}{\partial \xi} \right)^\alpha H_1(x, \Xi) ((1 + |\xi|^2)^{1/4} \zeta)^\alpha \right. \right. \\ & \left. \left. + \left(\frac{\partial}{\partial x} \right)^\alpha H_2(X, \xi) ((1 + |\xi|^2)^{-1/4} z)^\alpha \right] \right\}^k dz d\zeta \right]. \end{aligned}$$

Since θ is quadratic in (z, ζ) we can apply the lemma of stationary phase to conclude

$$c(x, \xi) = D(x, \xi) \left[a_0(x, \Xi(x, \xi)) b_0(X(x, \xi), \xi) + \sum_{l=1}^{\infty} \gamma_{-l}(x, \xi) \right], \quad (4.7)$$

where

$$D(x, \xi) = \det \begin{bmatrix} I & d_2^2 H_1(x, \Xi) \\ d_1^2 H_2(X, \xi) & I \end{bmatrix}^{-1/2} \in S^0$$

and $\gamma_{-l}(x, \xi) \in S^{-l}$ is a polynomial in

$$D_\xi^\alpha H_1(x, \Xi), D_x^\beta H_2(X, \xi), D_\xi^\beta a_{-j}(x, \Xi), D_x^\beta b_{-j}(x, \xi)$$

with $|\alpha| \leq 2(l+1)$ and $|\beta| + j \leq l$ (here $a(x, \xi) = \sum_{j=1}^{\infty} a_{-j}(x, \xi)$ and $b(x, \xi) = \sum_{j=0}^{\infty} b_{-j}(x, \xi)$, a_{-j} and b_{-j} homogeneous of degree $-j$). The coefficients of the polynomial are smooth functions of (x, ξ) , independent of H_1 and H_2 .

Thus the map $(a, b, H_1, H_2) \mapsto c(x, \xi)$ is continuous as a map

$$S^{0,k;l} \times S^{0,k;l} \times \mathcal{W}^t \times \mathcal{W}^t \rightarrow S^{0,k;l}$$

as long as $l \leq 2(t-k-1)$. The differentiability of the map $S^{0,k;\mathfrak{s}(t)} \times S^{0,k;\mathfrak{s}(t)} \times \mathcal{W}^{t+r} \times \mathcal{W}^{t+r} \rightarrow S^{0,k;\mathfrak{s}(t)}$, $\mathfrak{s}(t) = 2(t-k-1)$, is limited by the differentiability of the map $(H_1, H_2) \mapsto (X(x, \xi), \Xi(x, \xi))$ which involves the inverse function theorem and is thus only C^r . \square

Lemma 4.3. Let $\tilde{W} \subset \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O)$ be an open set with $\tilde{W}^{-1} \subset \mathcal{U}$ and let $\tilde{\mathcal{W}}$ denote $\Phi(\tilde{W}) \subset \mathcal{S}(\mathbb{R}^n \setminus O)$. For $\eta \in \tilde{W}$ let $H = \Phi(\eta)$ and $\tilde{H} = \Phi(\eta^{-1})$. Let $A \in \text{FIO}_{0,k}$ be given by

$$Au(x) = (2\pi)^{-n} \int \int e^{i((x-y) \cdot \xi + H(x, \xi))} u(y) dy d\xi$$

and define the symbol $\tilde{a}(x, \xi) \in S^{0,k}(\Omega \times \mathbb{R}^n)$ by

$$A^{-1}u(x) = (2\pi)^{-n} \int \int e^{i((x-y) \cdot \xi + \tilde{H}(x, \xi))} \tilde{a}(x, \xi) u(y) dy d\xi.$$

Then the map $\tilde{\mathcal{W}} \rightarrow S^{0,k}$ given by $H \mapsto \tilde{a}(x, \xi)$ is continuous. It extends to a C^r map $\tilde{\mathcal{W}}^{t+r} \rightarrow S^{0,k;\mathfrak{s}(t)}$.

Proof. Let $a_0(x, \xi) = \det \begin{bmatrix} I & d_2^2 H(x, \Xi) \\ d_1^2 \tilde{H}(X, \xi) & I \end{bmatrix}^{1/2}$ and set $B \in \text{FIO}_{0,k}$ by

$$Bu(x) = (2\pi)^{-n} \int \int e^{i((x-y) \cdot \xi + \tilde{H}(x, \xi))} a_0(x, \xi) u(y) dy d\xi.$$

By (4.7) $A \circ B$ is a pseudodifferential operator with principal symbol equal to 1, i.e. $A \circ B = I + R$, where $R \in \psi DO_{-1,k}$. If H is small then the symbol, $r(x, \xi)$, of R is small so the Calderón-Vaillancourt theorem says the inverse $(I + R)^{-1}$ exists in $\psi DO_{0,k}$. Thus $A^{-1} = B(I + R)^{-1}$ and by Lemma 4.2 and the fact inversion is C^r from $(\psi DO_{0,k}^{s+r})_* \rightarrow \psi DO_{0,k}^s$ we get the lemma. \square

Now from Propositions 3.1 and 4.1 we conclude

Theorem 4.4. *The group $(\text{FIO}_{0,k})_*$ is a topological group such that the maps i and π in the exact sequence (4.1) are continuous and π is open. Furthermore the local section σ (4.2) is continuous.*

Now let $\mathcal{H}^s = (\psi DO_{0,k})_*^s$ and $\mathcal{Q}^t = (\mathcal{D}_\theta(T^*M \setminus O))^t$ be the spaces $(\psi DO_{0,k})_*$ and $\mathcal{D}_\theta(T^*M \setminus O)$ with the coarser topologies of $H^s = (\psi DO_{0,k}^s)_*$ and $\mathcal{Q}^t = \mathcal{D}_\theta^t(T^*M \setminus O)$ respectively. It follows from the lemmas above, that if $t > 2n$ and $s(t) = t - 2(k+1) > n$ then the map

$$(A') \quad V^t \times V^t \times (\psi DO_{0,k})_*^{s(t)} \rightarrow (\psi DO_{0,k})_*^{s(t)} \\ (\eta_1, \eta_2, P) \mapsto \sigma(\eta_1) \sigma(\eta_2)^{-1} P \sigma(\eta_1 \eta_2^{-1})^{-1}$$

is continuous (in fact C^r). Applying Proposition 3.1 for each t , we make $(\text{FIO}_{0,k})_*$ a topological group $(\text{FIO}_{0,k})_*^t$ so that

$$I \rightarrow (\psi DO_{0,k})_*^{s(t)} \rightarrow (\text{FIO}_{0,k})_*^t \rightarrow (\mathcal{D}_\theta(T^*M \setminus O))^t \rightarrow e$$

is an exact sequence of topological groups for each $t > t_0 = \max\{2n, n + 2(k+1)\}$. Let $(\text{FIO}_{0,k}^t)_*$ denote the completion of $(\text{FIO}_{0,k})_*^t$ with respect to the right uniform structure as described in Sect. 3, i.e. setting $G^t = (\text{FIO}_{0,k}^t)_*$. To make sure that $(\text{FIO}_{0,k}^t)_*$ is a topological group we must verify condition (A'') of Sect. 3, i.e. the local section σ extends to a continuous local section $\tilde{\sigma}: \tilde{U}^t \rightarrow (\text{FIO}_{0,k})_*$ and the map (A') extends to a continuous map

$$(A'') \quad \tilde{V}^t \times \tilde{V}^t \times (\psi DO_{0,k}^{s(t)})_* \rightarrow (\psi DO_{0,k}^{s(t)})_*.$$

If $\eta \in \tilde{U}^t$ we can define $\tilde{\sigma}(\eta) \in (\text{FIO}_{0,k}^t)_*$ as represented by a Fourier integral operators of order zero with symbol $a(x, \xi) = \sum_{j=0}^k a_{-j}(x, \xi)$ with $a_{-j} \in H^{s(t)+k-j}(T^*M \setminus O)$, and phase function φ_H with generating function $\eta_H \in \mathcal{D}_\theta^t(T^*M \setminus O)$ i.e. $H \in \mathcal{S}^t(T^*M \setminus O)$. Then the operator defined by (2.6) still makes sense as an oscillatory integral if we can differentiate $a(x, \xi)$ and $H(x, \xi)$ enough times so that they become integrable on $T^*M \setminus O$. If $t > 2n$ the first n derivatives of $a(x, \xi)$ and $H(x, \xi)$ are continuous, so integration by parts n -times gives a convergent expression. Therefore the local section extends a continuous map $\tilde{\sigma}$. From the corollaries to the lemmas above we can conclude that condition (A') extends to a continuous map (A'') if $t > 2n$ and $s(t) > n$. We conclude

Theorem 4.5. *Let $t > 2n$ and $s(t) = t - 2(k+1) > n$. Then*

$$I \longrightarrow (\psi DO_{0,k}^{s(t)})_* \xrightarrow{j} (\text{FIO}_{0,k}^t)_* \xrightarrow{\pi} \mathcal{D}_\theta^t(T^*M \setminus O) \rightarrow e$$

is an exact sequence of topological groups, such that j and π are continuous and π is open. Furthermore (2.7) defines a continuous local section

$$\tilde{\sigma}: \mathcal{U}^t \subset \mathcal{D}_\theta^t(T^*M \setminus O) \rightarrow (\text{FIO}_{0,k}^t)_*.$$

In order to define a Hilbert manifold structure on $(\text{FIO}_{0,k}^t)_*$ we need to verify condition (C) in Sect. 3, the overlap condition for two charts:

Proposition 4.6. *Let $t > 2n$ and $s(t) = t - 2(k+1) > n$. For $A_0, \tilde{A}_0 \in (\text{FIO}_{0,k}^t)_*$ let $\pi(A_0) = \alpha_0$ and $\pi(\tilde{A}_0) = \tilde{\alpha}_0$. Then condition (C) is satisfied:*

(C) *The map*

$$U^t \cdot \alpha_0 \cap U^t \cdot \tilde{\alpha}_0 \times (\psi DO_{0,k}^{s(t)})_* \rightarrow (\psi DO_{0,k}^{s(t)})_* \\ (\eta, P) \mapsto P \sigma(\eta \alpha_0^{-1}) A_0 \tilde{A}_0^{-1} \sigma(\eta \tilde{\alpha}_0^{-1})^{-1}$$

is C^t .

Proof. Since multiplication in $(\psi DO_{0,k}^s)_*$ is smooth we need only show that

$$\eta \mapsto \sigma(\eta \alpha_0^{-1}) A_0 \tilde{A}_0^{-1} \sigma(\eta \tilde{\alpha}_0^{-1})^{-1}$$

is C^t .

Let $A = A_0 \tilde{A}_0^{-1}$ and $\alpha = \pi(A)$, then it is enough to see that

$$\eta \mapsto \sigma(\eta) A \sigma(\eta \alpha)^{-1}$$

is C^t for η and $\eta \alpha$ near the identity. This implies that α is also near the identity. Then the claim follows from Lemmas 4.2 and 4.3. \square

We have the following

Theorem 4.7. *If $t > 2n$ and $s(t) = t - 2(k+1) > n$, then $(\text{FIO}_{0,k}^t)_*$ is a C^t -differentiable Hilbert manifold modelled on the space $\mathcal{S}^{t+1}(T^*M \setminus O) \times (\psi DO_{0,k}^{s(t)})_*$.*

Remark. Since the manifold structure on $(\text{FIO}_{0,k}^t)_*$ is defined by right translation of a chart about the identity, it is automatic that right translation in $(\text{FIO}_{0,k}^t)_*$ is C^t .

To finish the construction of an ILH-Lie group structure for $(\text{FIO}_{0,k}^t)_*$ we need to verify condition (D) of Sect. 3, which guarantees that multiplication is C^l .

Proposition 4.8. *Let $t > 2n$ and $s(t) = t - 2(k+1) > n$. For $A \in (\text{FIO}_{0,k}^{t+})_*$, $B \in (\text{FIO}_{0,k}^t)_*$, $\alpha = \pi(A)$, $\beta = \pi(B)$, condition (D) is satisfied:*

(D) *The map*

$$\mathcal{U}_\alpha^{t+r} \times \mathcal{U}_\beta^t \times (\psi DO_{0,k}^{s(t)})_* \rightarrow (\psi DO_{0,k}^{s(t)})_* \\ (\eta_1, \eta_2, P) \mapsto \sigma(\eta_1, \alpha^{-1}) A P \sigma(\eta_2 \beta^{-1}) A^{-1} \sigma(\eta_1 \eta_2 \beta^{-1} \alpha^{-1})^{-1}$$

is C^l for $l = \min(r, t)$.

Proof. This follows from the fact that the map (A') (and hence (B')) is actually C^t (Lemmas 4.2 and 4.3), and that multiplication in $\mathcal{D}_\theta(T^*M \setminus O)$ is C^r as a map $\mathcal{D}_\theta^{t+r}(T^*M \setminus O) \times \mathcal{D}_\theta^t(T^*M \setminus O) \rightarrow \mathcal{D}_\theta^t(T^*M \setminus O)$. \square

In summary we have the following

Theorem 4.9. $\{(\text{FIO}_{0,k})_*, (\text{FIO}_{0,k}^t)_* \mid t > \max\{2n, n + 2(k+1)\}\}$ is an ILH-Lie group with

(i) $(\text{FIO}_{0,k}^t)_*$ is a C^t -Hilbert manifold modelled on

$$\mathcal{S}^{t+1}(T^*M \setminus O) \times (\psi D O_{0,k}^{t-2(k+1)})_*.$$

The inclusion

$$(\text{FIO}_{0,k}^{t+1})_* \hookrightarrow (\text{FIO}_{0,k}^t)_* \quad \text{is } C^t.$$

(ii) The group multiplication $(\text{FIO}_{0,k})_* \times (\text{FIO}_{0,k})_* \rightarrow (\text{FIO}_{0,k})_*$ extends to a C^l map

$$(\text{FIO}_{0,k}^{t+r})_* \times (\text{FIO}_{0,k}^t)_* \rightarrow (\text{FIO}_{0,k}^t)_* \\ l = \min(r, t).$$

(iii) The inversion $(\text{FIO}_{0,k})_* \rightarrow (\text{FIO}_{0,k})_*$ extends to a C^l map

$$(\text{FIO}_{0,k}^{t+r})_* \rightarrow (\text{FIO}_{0,k}^t)_* \\ l = \min(r, t).$$

(iv) Right multiplication for $A \in (\text{FIO}_{0,k}^t)_*$ is a C^t map $R_A: (\text{FIO}_{0,k}^t)_* \rightarrow (\text{FIO}_{0,k}^t)_*$.

5. Conclusions and Remarks

1) In Sect. 4 we have given $(\text{FIO}_{0,k})_*$ an ILH-Lie group structure. Since we began the paper by asking about the full group of invertible formal Fourier integral operators we should make some remarks in that respect. First, consider $(\text{FIO}_0)_*$, we have

Corollary 5.1. $(\text{FIO}_0)_*$ has the Lie group structure of a direct limit of ILH-Lie groups, i.e. $(\text{FIO}_0)_* = \varinjlim (\text{FIO}_{0,k})_*$.

We should probably call this a DLILH-Lie group but we will try to refrain from this monstrosity.

The situation for the full group FIO_* is even more complicated. For any m we can give $\text{FIO}_m \cap \text{FIO}_*$ the structure of a direct limit of ILH-manifolds by using an elliptic operator to give an identification of $\text{FIO}_m \cap \text{FIO}_*$ with $(\text{FIO}_0)_*$. Multiplication will be smooth in an appropriate sense between the appropriate spaces. Piecing this together for all m makes FIO_* a *graded direct limit of ILH-Lie groups*. We omit precise formulations of these concepts since they are direct generalizations of Sect. 3.

It should also be noted that one can tack on the smoothing operators as described in [16]. Let $(F_0)_*$ denote the group of invertible classical Fourier integral operators. Then $(\text{FIO}_0)_*$ is the quotient of $(F_0)_*$ by the smoothing operators. One can easily find a section $\sigma: (\text{FIO}_0)_* \rightarrow (F_0)_*$ (see Lemma 5.7 of paper I in [16]) so we can write $(F_0)_*$ as the product $(\text{FIO}_0)_* \times C^\infty(M \times M)$. In [16,

paper II], it is shown that $(F_0)_*$ is a topological group when modelled in this way on $(FIO_0)_* \times C^\infty(M \times M)$. This product also makes $(F_0)_*$ into a Fréchet manifold but at this point we have been unable to check that $(F_0)_*$ is a Fréchet-Lie group.*

2) We can identify the ILH-Lie algebras of our ILH-Lie groups. We consider here $(FIO_{0,k})_*$, the situation for $(FIO_0)_*$ and FIO_* is, of course, similar.

Theorem 5.2. *The Lie algebra of $(FIO_{0,k}^s)_*$ is the space $\tilde{\psi}DO_{1,k}^s$ of H^s formal pseudodifferential operators of order one modulo those of order $-k-1$ with pure imaginary principal symbol. The Lie bracket corresponds to the commutator bracket.*

Proof. Let $\alpha(t)$ be a C^1 curve in $(FIO_{0,k}^s)_*$ with $\alpha(0) = I$. Since the local section σ is smooth, we can write $\alpha(t) = P(t)\sigma(\pi(\alpha(t)))$ where $P(t)$ is a C^1 curve in $\psi DO_{0,k}^s$ with $P(0) = I$ and $\sigma(\pi(\alpha(t)))$ has the local expression

$$\sigma(\pi(\alpha(t)))u(x) = (2\pi)^{-n} \int \int e^{i(\varphi_0(x,y,\xi) + H_t(x,\xi))} u(y) dy d\xi,$$

where $\varphi_0(x, y, \xi) = (x - y) \cdot \xi$, $H_0(x, \xi) = 0$ and $H_t(x, \xi) \in \mathcal{S}^{s+k+1}$ is C^1 in t .

Differentiating $\alpha(t)$ with respect to t at $t=0$ gives

$$\alpha'(0) = P'(0) + A,$$

where A has the local expression

$$Au(x) = (2\pi)^{-n} \int \int e^{i\varphi_0(x,y,\xi)} iH'_0(x,\xi)u(y) dy d\xi,$$

where $H'_0(x, \xi) = \frac{d}{dt} H_t(x, \xi)|_{t=0}$. Since $H_t(x, \xi)$ is homogeneous of degree one in ξ for each t , $H'_0(x, \xi)$ is homogeneous of degree one in ξ . Also $H'_0(x, \xi) \in \mathcal{S}^{s+k+1}(T^*M \setminus O)$ so $A \in \tilde{\psi}DO_{1,k}^s$ and $P'(0) \in \psi DO_{0,k}^s$ so $\alpha'(0) \in \tilde{\psi}DO_{1,k}^s$. It is now straightforward to check that the Lie bracket is indeed the commutator bracket. \square

It is easy to see that the Lie algebra of $(\psi DO_{0,k}^s)_*$ is $\psi DO_{0,k}^s$, again with the commutator bracket, and it is shown in [18] that the Lie algebra of $\mathcal{D}_\theta^{s+k}(T^*M \setminus O)$ is the space $\mathcal{S}^{s+k+1}(T^*M \setminus O)$ with Poisson bracket. Thus we have

Proposition 5.3. *The exact sequence of ILH-Lie groups*

$$I \longrightarrow (\psi DO_{0,k}^s)_* \longrightarrow (FIO_{0,k}^s)_* \xrightarrow{\pi} \mathcal{D}_\theta^{s+k}(T^*M \setminus O) \longrightarrow e \quad (5.1)$$

has the corresponding sequence of ILH-Lie algebras

$$0 \longrightarrow \psi DO_{0,k}^s \longrightarrow \tilde{\psi}DO_{1,k}^s \xrightarrow{q} \mathcal{S}^{s+k+1}(T^*M \setminus O) \longrightarrow 0. \quad (5.2)$$

The map $q = T_I \pi$ is $\frac{1}{i}$ times the principal symbol.

Remark. One can check directly that q is a Lie algebra homomorphism since if $P_1, P_2 \in \psi DO_{1,k}^s$ with principal symbols $ip_1(x, \xi)$ and $ip_2(x, \xi)$, then the principal symbol of $[P_1, P_2]$ is $\frac{1}{i} \{ip_1(x, \xi), ip_2(x, \xi)\} = i\{p_1(x, \xi), p_2(x, \xi)\}$.

* The recent papers [16a, VII, VIII] show that this is a regular Fréchet Lie group

Having described the Lie algebras we can give the exponential maps. Indeed, the map $\exp_1 : \psi DO_{0,k}^s \rightarrow (\psi DO_{0,k}^s)_*$ is just exponentiation of operators (modulo equivalence classes). This is well defined since the exponential of a zeroth order classical pseudodifferential operator is also a zeroth order classical pseudodifferential operator. The map $\exp_2 : \tilde{\psi} DO_{1,k}^s \rightarrow (\text{FIO}_{0,k}^s)_*$ is also exponentiation of operators. To see this is well defined take $P \in \tilde{\psi} DO_{1,k}^s$, since P has purely imaginary principal symbol the solution $U(t)$ to $\left(\frac{d}{dt} + P\right)U(t) = 0$ with $U(0) = I$ is a one parameter family of Fourier integral operators of order zero. Finally, $\exp_3 : \mathcal{S}^{s+k+1} \rightarrow \mathcal{D}_\theta^{s+k}(T^*M \setminus O)$ is given by $\exp_3(tH) = \eta_t$ where η_t is the flow of the Hamiltonian vector field X_H on $T^*M \setminus O$, see [18].

If we put the exact sequences (5.1) and (5.2) together by means of the exponential maps we get

Proposition 5.4. *The following diagram commutes*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \psi DO_{0,k}^s & \longrightarrow & \tilde{\psi} DO_{1,k}^s & \xrightarrow{e} & \mathcal{S}^{s+k+1}(T^*M \setminus O) \longrightarrow 0 \\ & & \downarrow \exp_1 & & \downarrow \exp_2 & & \downarrow \exp_3 \\ I & \longrightarrow & (\psi DO_{0,k}^s)_* & \longrightarrow & (\text{FIO}_{0,k}^s)_* & \xrightarrow{\pi} & \mathcal{D}_\theta^{s+k}(T^*M \setminus O) \longrightarrow e. \end{array}$$

Remark. A) The second square commutes by Ergorov's theorem. Namely, if $U(t)$ satisfies $\left(\frac{d}{dt} + P\right)U(t) = 0$, $U(0) = I$ for $P \in \tilde{\psi} DO_{1,k}^s$ then the canonical transformation of $U(t)$ is the flow of the Hamiltonian vector field $X_{\rho(P)}$ on $T^*M \setminus O$.

B) As for any diffeomorphism group, the exponential map $\exp_3 : \mathcal{S}^{s+1}(T^*M \setminus O) \rightarrow \mathcal{D}_\theta^s(T^*M \setminus O)$ is not onto a neighborhood of the identity, i.e. it cannot be used as a local chart. As a consequence, the same is true for \exp_2 .

3) Throughout this paper we've studied the exact sequence (5.1). It is natural to ask if this sequence splits. It is easy to see that such a splitting would make $(\text{FIO}_{0,k}^s)_*$ a semidirect product of $(\psi DO_{0,k}^s)_*$ and $\mathcal{D}_\theta(T^*M \setminus O)$. This structure would make most of the work in this paper much easier. To see that the sequence (5.1) does not split it is enough to show that the sequence of Lie algebras (5.2) doesn't split. In paper III of [16] it is shown in fact that (5.2) does split for $k=0$ and does not split for $k=1$. It has been pointed out by Duistermaat [9] that a splitting for $k=0$ is given by mapping $a \in \mathcal{S}(T^*M \setminus O)$ to $A \in \tilde{\psi} DO_1 / \psi DO_{-1}$ with principal symbol ia and subprincipal symbol zero. Let $\mathcal{B} \subset \tilde{\psi} DO_1$ be the subalgebra of operators with vanishing subprincipal symbol. Then a splitting of (5.2) for $k=1$ would mean a splitting of the sequence

$$0 \rightarrow \psi DO_{-1} / \psi DO_{-2} \rightarrow \mathcal{B} / \psi DO_{-2} \rightarrow \mathcal{S} \rightarrow 0.$$

In [16] it is shown that this sequence never splits, when $n \geq 2$.

We could also ask if the sequence (5.1) splits topologically, i.e., what is the topology of the principal bundle

$$\begin{array}{ccc} (\psi DO_{0,k}^s)_* & \longrightarrow & (\text{FIO}_{0,k}^s)_* \\ & & \downarrow \\ & & \mathcal{D}_\theta(T^*M \setminus O). \end{array}$$

Again, we expect this bundle to be nontrivial but we have no proof of this.

4) It is interesting to ask if the group of all zeroth order invertible Fourier integral operators \mathcal{F}_0^* can be made into an ILH-Lie group by our methods. By this we mean full operators (not equivalence classes) with general (not necessarily classical) symbols. Again there is an exact sequence

$$I \rightarrow \mathcal{P}_0^* \rightarrow \mathcal{F}_0^* \rightarrow \mathcal{D}_\theta(T^*M \setminus O) \rightarrow e, \quad (5.3)$$

where \mathcal{P}_0^* is the full group of invertible zeroth order pseudodifferential operators. \mathcal{P}_0^* is a topological group using the seminorm topology on symbol spaces (and by choosing a global phase function). The first difficulty in following through our program arises in finding the local section $\sigma: \mathcal{U} \subset \mathcal{D}_\theta(T^*M \setminus O) \rightarrow \mathcal{F}_0^*$. The values of the section we use for $(\text{FIO}_{0,k})_*$ may not be actually invertible, they are however invertible modulo smoothing operators. This problem can be dealt with by arranging that $\sigma(e) = I$ (not just modulo smoothing), then nearby $\sigma(\eta)$ must still be invertible.

A more serious problem rises in trying to define the spaces $(\mathcal{P}_0^s)_*$. The quotient spaces $\mathcal{P}_{0,k}$ are not nice for \mathcal{P}_0 since the space \mathcal{P}_{-k} are not closed in \mathcal{P}_0 . The most likely candidates for the spaces \mathcal{P}_0^s would be Banach spaces defined by the local norms

$$\|P\|_s^2 = \sum_{k=0}^s \|[X_{i_1}, [X_{i_2}, \dots [X_{i_k}, P] \dots]]\|_0^2,$$

where X_1, \dots, X_n are fixed local basis of vector fields and $\|\cdot\|_0$ the L^2 -operator norm, see Beals [4], Coifman-Meyer [7].

Using these topologies and a C^k version of $\mathcal{D}_\theta^s(T^*M \setminus O)$ one should be able to carry out the program outlined in this paper for the exact sequence (5.3).

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